AFOSR TN-57-419
ASTIA DOCUMENT No. AD 132498

INSTITU[illegible] [illegible]NCES
25 Waverly [illegible] [illegible] Y.

NEW YORK UNIVERSITY
Institute of Mathematical Sciences
Division of Electromagnetic Research

NEW YORK UNIV[illegible]ITY
[illegible]TE OF MAT[illegible] [illegible]CIENCES
L[illegible]
25 Waverly Place, New York 3, N.Y.

RESEARCH REPORT No. BR-23

Expansions for Products of Two Whittaker Functions

F. M. RAGAB

MATHEMATICS DIVISION
AIR FORCE OFFICE OF SCIENTIFIC RESEARCH
CONTRACT No. AF 18(600)-367
FILE No. 2.2

JULY, 1957

NEW YORK UNIVERSITY

Institute of Mathematical Sciences
Division of Electromagnetic Research

Research Report No. BR-23

EXPANSIONS FOR PRODUCTS OF TWO WHITTAKER FUNCTIONS

F. M. Ragab

F. M. Ragab
F. M. Ragab

Morris Kline
Morris Kline
Project Director

July, 1957

Qualified requestors may obtain copies of this report from the ASTIA Document Service Center, Dayton 2, Ohio. Department of Defense contractors must be established for ASTIA services, or have their 'need-to-know' certified by the cognizant military agency of their project or contract.

New York, 1957

Table of Contents

		Page
1.	Introduction	1
2.	Expansions of E-functions	4
3.	Linear relations between E-functions	6
4.	Duplication and addition formulas for the E-functions	8
5.	A certain kind of transformation between generalized hypergeometric functions of arguments $\pm \frac{1}{2}$ and ± 1	11
	References	13
	Distribution List	

Abstract

Whittaker functions and E-functions are expanded in terms of products of these functions. Also, products of two Whittaker functions or two E-functions are expanded, in series involving these functions. The variable is the same in all terms of the series and the values of the parameters differ by integers.

1. Introduction

We shall prove a number of sum formulas for series involving MacRobert's E-function and its special case, the Whittaker function, which plays a role in the theory of diffraction by a paraboloid of revolution.

The expansions to be proved are:

$$(1)\qquad \sum_{r=0}^{\infty} \frac{(b;r)(b';r)(a;r)}{r!} W_{\frac{1}{2}(1-a-b-2r),\frac{1}{2}(b-a)}(x) W_{\frac{1}{2}(1-a-b'-2r),\frac{1}{2}(b'-a)}(x)$$

$$= e^{-\frac{1}{2}x} x^{\frac{1}{2}-\frac{1}{2}a} W_{\frac{1}{2}(1-a-b-b'),\ \frac{1}{2}(b+b'-a)}(x),$$

$$(2)\qquad \sum_{r=0}^{\infty} \frac{(-1)^r(a;r)(b;r)(b';r)}{r!} x^{-\frac{1}{2}r} W_{\frac{1}{2}(1-a-b-b'-3r),\ \frac{1}{2}(b+b'-a+r)}(x)$$

$$= e^{\frac{1}{2}x} x^{\frac{1}{2}a-\frac{1}{2}} W_{\frac{1}{2}(1-a-b),\ \frac{1}{2}(b-a)}(x) W_{\frac{1}{2}(1-a-b'),\ \frac{1}{2}(b'-a)}(x),$$

$$(3)\qquad \sum_{r=0}^{\infty} \frac{(a;r)\{(b;r)\}^2}{r!} \cdot \left\{ W_{\frac{1}{2}(1-a-b-2r),\ \frac{1}{2}(b-a)}(x) \right\}^2$$

$$= e^{-\frac{1}{2}x} x^{\frac{1}{2}-\frac{1}{2}a} W_{\frac{1}{2}(1-a-2b),\ \frac{1}{2}(2b-a)}(x),$$

$$(4)\qquad \sum_{r=0}^{\infty} \frac{(-1)^r(a;r)\{(b;r)\}^2}{r!} x^{-\frac{1}{2}r} W_{\frac{1}{2}(1-a-2b-3r),\ \frac{1}{2}(2b-a+r)}(x)$$

$$= e^{\frac{1}{2}x} x^{\frac{1}{2}a-\frac{1}{2}} \left\{ W_{\frac{1}{2}(1-a-b),\ \frac{1}{2}(b-a)}(x) \right\}^2,$$

$$(5)\qquad \sum_{r=0}^{\infty} \frac{(-\frac{1}{2};r)(a;r)(b;r)}{r!} \left\{ W_{\frac{1}{2}(1-a-b-2r),\ \frac{1}{2}(b-a)}(x) \right\}^2$$

$$= 2^{a+b-\frac{1}{2}} x^{\frac{1}{2}} W_{(\frac{1}{2}-a-b),(b-a)}(2x),$$

$$(6)\qquad \sum_{r=0}^{\infty} \frac{(-\frac{1}{2};r)(a+1;r)(b;r)}{r!} W_{\frac{1}{2}(1-a-b-2r),\, \frac{1}{2}(b-a)}(x) W_{\frac{1}{2}(-a-b-2r),\, \frac{1}{2}(b-a-1)}(x)$$

$$= 2^{a+b} x^{\frac{1}{2}} W_{(-a-b),(b-a-\frac{1}{2})}(2x),$$

$$(7)\qquad \sum_{r=0}^{\infty} \frac{(\frac{1}{2};r)(a;r)(b;r)}{r!} 2^{2r} W_{\frac{1}{2}(1-2a-2b-4r),\, \frac{1}{2}(2b-2a)}(2x)$$

$$= 2^{-\frac{1}{2}(2a+2b-2)} x^{-\frac{1}{2}} \left\{ W_{\frac{1}{2}(1-a-b),\, \frac{1}{2}(b-a)}(x) \right\}^2 ,$$

and

$$(8)\qquad \sum_{r=0}^{\infty} \frac{(\frac{1}{2};r)(a+1;r)(b;r)}{r!} 2^{2r} W_{\frac{1}{2}(-2a-2b-4r),\, \frac{1}{2}(2b-2a-1)}(2x)$$

$$= 2^{-\frac{1}{2}(2a+2b)} x^{-\frac{1}{2}} W_{\frac{1}{2}(1-a-b),\, \frac{1}{2}(b-a)}(x) W_{\frac{1}{2}(-a-b),\, \frac{1}{2}(b-a-1)}(x) .$$

They will be established by means of new expansions of MacRobert's E-functions to be proved in Sections 2 and 4. For the definitions and properties of the E-functions, see MacRobert [1].

In Section 3 of this paper, a new linear relation for the E-function will be established. The sum of an infinite series, involving generalized hypergeometric functions of argument -1, is also obtained.

In Section 5, certain relations between generalized hypergeometric functions of argument +1 and $+\frac{1}{2}$ will be obtained.

The following formulas are required in the proofs:

$$(9)\qquad E(\alpha,\beta::z) = \Gamma(\alpha) \int_0^{\infty} e^{-\lambda} \lambda^{\beta-1} \left(1 + \frac{\lambda}{z}\right)^{-\alpha} d\lambda,$$

where $R(\beta) > 0$, $|\text{amp } z| < \pi$, (see [1], p. 348);

$$(10) \qquad \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} E(a,b+s::x)E(a',b'-s::x)ds$$

$$= 2^{-b-b'}\frac{\Gamma(a)\Gamma(a')}{\Gamma(a+a')} E(a+a', b+b' :: 2x),$$

(see [2]).

Other formulas required are:

$$(11) \qquad \int_0^\infty e^{-t}\, t^{\gamma-1}\, E(\alpha,\beta::tz)dt = \Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\gamma)\Gamma(\beta+\gamma)\left\{\Gamma(\alpha+\beta+\gamma)\right\}^{-1}$$

$$\cdot F(\alpha,\beta;\alpha+\beta+\gamma;\frac{z-1}{z}),$$

if $R(z) > \frac{1}{2}$, $R(\alpha+\gamma) > 0$, $R(\beta+\gamma) > 0$, (see [1], p. 396, ex. 113). If $R(z) > \frac{1}{2}$, $R(\alpha+\gamma+\lambda) > 0$, $R(\beta+\gamma+\lambda) > 0$, $R(\alpha+\gamma+\mu) > 0$, $R(\beta+\gamma+\mu) > 0$ then

$$(12) \qquad \int_0^\infty e^{-t}\, t^{\gamma-1}\, E(\alpha,\beta::t)E(\lambda,\mu::zt)dt = \Gamma(\beta)\Gamma(\lambda)\Gamma(\alpha+\gamma+\lambda)\Gamma(\beta+\gamma+\mu)$$

$$\cdot \sum_{r=0}^{\infty} \frac{\Gamma(\alpha+r)\Gamma(\mu+r)\Gamma(\alpha+\gamma+\mu+r)}{r!\,\Gamma(\alpha+\beta+\gamma+\mu+r)\Gamma(\alpha+\gamma+\lambda+\mu+r)}\, F\left(\begin{matrix}\lambda,\ \mu+r;\ \frac{z-1}{z} \\ \alpha+\gamma+\lambda+\mu+r\end{matrix}\right),$$

(see [1], p. 397, ex. 117);

$$(13) \qquad E(\alpha,\beta::z) = \Gamma(\alpha)\Gamma(\beta)\, z^{\frac{1}{2}(\alpha+\beta-1)}\, e^{\frac{1}{2}z}\, W_{\frac{1}{2}(1-\alpha-\beta),\, \frac{1}{2}(\beta-\alpha)}(z),$$

(see [1], p. 351).

2. Expansions of E-functions

The two formulas to be proved here are:

$$(14)\qquad \sum_{r=0}^{\infty} \frac{x^{-2r}}{r!\,\Gamma(a+r)}\, E(a+r,b+r::x)\, E(a+r,b'+r::x)$$

$$= \frac{\Gamma(b)\Gamma(b')}{\Gamma(b+b')}\, E(a,b+b'::x),$$

and

$$(15)\qquad \sum_{r=0}^{\infty} \frac{(-1)^r\Gamma(b+r)\Gamma(b'+r)}{r!\,\Gamma(b+b'+2r)}\, x^{-2r}\, E(a+r,b+b'+2r::x)$$

$$= \frac{1}{\Gamma(a)}\, E(a,b::x)\, E(a,b'::x)$$

To prove (14), substitute for each E-function on the left-hand side of (14) from (9) and change the order of integration and summation, thus finding

$$\text{l.h.s. of (14)} \quad = \Gamma(a)\int_0^\infty\int_0^\infty e^{-(\lambda+\mu)}\,\lambda^{b-1}\mu^{b'-1}\,d\lambda d\mu$$

$$\cdot\left[(1+\frac{\lambda}{x})(1+\frac{\mu}{x})\right]^{-a}\quad {}_1F_0\left[a;\,\frac{\lambda\mu}{(\lambda+x)(\mu+x)}\right]$$

$$= \Gamma(a)\int_0^\infty\int_0^\infty e^{-(\lambda+\mu)}\,\lambda^{b-1}\mu^{b'-1}\left[1+\frac{\lambda+\mu}{x}\right]^{-a} d\lambda d\mu.$$

Here put $\lambda = \mu\zeta$ where ζ is a new variable of integration; then

$$\text{l.h.s. of (14)} \quad = \Gamma(a)\int_0^\infty\int_0^\infty e^{-\mu(1+\zeta)}\,\zeta^{b-1}\,\mu^{b+b'-1}\left[1+\frac{\mu(1+\zeta)}{x}\right]^{-a} d\mu d\zeta.$$

Now substitute $\mu = \frac{\lambda}{1+\zeta}$ and change the order of integration so that the integral with respect to ζ becomes the last, then

$$\text{l.h.s. of (14)} \quad = \Gamma(a) \int_0^\infty e^{-\lambda} \lambda^{b+b'-1} \left(1+\frac{\lambda}{x}\right)^{-a} d\lambda \int_0^\infty \frac{\zeta^{b-1}}{(1+\zeta)^{b+b'}} d\zeta$$

$$= \frac{\Gamma(b)\Gamma(b')}{\Gamma(b+b')} E(a,b+b'::x) \;, \quad \text{by (9).}$$

To prove (15), substitute from (14) for the E-function on the left-hand side of (15), with the result that

$$\text{l.h.s. of (15)} \quad = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} x^{-2r} \sum_{s=0}^{\infty} \frac{x^{-2s}}{s!\,\Gamma(a+r+s)}$$

$$\cdot\, E(a+r+s,b+r+s::x)\, E(a+r+s,b'+r+s::x).$$

Here take p-r as the parameter of summation and, after some rearrangement, find

$$\text{l.h.s. of (15)} \quad = \sum_{p=0}^{\infty} \frac{x^{-2p}}{p!\,\Gamma(a+p)} E(a+p,b+p::x)E(a+p,b'+p::x)$$

$$\cdot \left\{ 1 + (-p) + \frac{(-p)(-p+1)}{2!} + \cdots \right\} .$$

But the expression between the brackets $\left\{ \quad \right\}$ vanishes unless $p = 0$. The outer sum therefore reduces to the first term

$$\left\{ \Gamma(a) \right\}^{-1} E(a,b::x)E(a,b'::x),$$

and the proof of (15) is complete.

The formulas (1), (2), (3), (4) can be deduced from (14) and (15) by means of (13) with $b = b'$ for (3) and (4).

3. Linear relations between E-functions

In (14), write b+s for b and b'-s for b' and integrate both sides of (14) applying (10) and the formula ([3], p. 193),

$$(16) \qquad \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \Gamma(b+s)\Gamma(b'-s) = 2^{-(b+b')}\,\Gamma(b+b'),$$

thus obtaining

$$(17) \qquad \sum_{r=0}^{\infty} \frac{(2x)^{-2r}\,2^{-2r}}{r!\,\Gamma(a+r+\frac{1}{2})}\,E(2a+2r,2r+b+b'::2x) = \frac{2^{a-1}}{\sqrt{\pi}}\,E(a,b+b'::x),$$

which can be generalized in the form

$$(18) \qquad \sum_{r=0}^{\infty} \frac{1}{r!\,\Gamma(\frac{1}{2}\,\alpha_t + r + \frac{1}{2})}\,(2x)^{-2r}\,E(p;\alpha_t + 2r:q;\,\rho_\sigma + 2r:x)$$

$$= 2^{\alpha_t - 1}\,\pi^{-\frac{1}{2}}\,E(\alpha_1,\alpha_2,\ldots,\tfrac{1}{2}\,\alpha_t,\ldots,\alpha_p:q;\rho_\sigma:\tfrac{x}{2}),$$

where $t = 1, 2, 3, \ldots, p$, and $p \geq 1$.

It may be noted that if $p = q = 1$ and $\rho_1 = \alpha_1$ in (18), we obtain the known expansion

$$(19) \qquad {}_1F_1(a;2a;2x) = e^{x}\;{}_0F_1(;\,a + \tfrac{1}{2}\,;\,x^2/4),$$

from which the known expansion of the modified Bessel function of the first kind, namely

$$(20) \qquad I_n(z) = \left\{\Gamma(n+1)\right\}^{-1}(\tfrac{1}{2}\,z)^n\,e^{-z}\;{}_1F_1(n+\tfrac{1}{2}\,,\,2n+1;2z),$$

is deduced.

Again, in (15), write b+s for b and b'-s for b'; integrate both sides of (15) with respect to s, applying (10) and (16), and get

$$(21) \qquad \sum_{r=0}^{\infty} \frac{(-1)^r (2x)^{-2r}}{r!} E(a+r, b+b'+2r::x) = \frac{\Gamma(a)}{\Gamma(2a)} E(2a, b+b'::2x),$$

which can be generalized in the form

$$(22) \qquad \sum_{s=0}^{\infty} \frac{(-1)^s x^{-2s}}{s!} E(p;\alpha_r+2s, b+s:q;\rho_t+2s: \tfrac{1}{2} x)$$

$$= \frac{\Gamma(b)}{\Gamma(2b)} E(p;\alpha_r, 2b:q;\rho_t::x).$$

We shall now sum certain infinite series of generalized hypergeometric functions of argument -1. For this purpose, the following formulas are required ([4], p. 194):

$$(23) \qquad \int_0^{\infty} \lambda^{-\alpha_{p+1}-1} E(p;\alpha_r:q;\rho_s;\lambda) E(\ell;\beta_t:m;\sigma_u:z\lambda)\, d\lambda$$

$$= \pi^{p-q} z^{\alpha_{p+1}} \sum_{r=1}^{p+1} \prod_{t=1}^{q} \sin(\rho_t-\alpha_r)\pi \left\{ {\prod_{s=1}^{p+1}}' \sin(\alpha_s-\alpha_r)\pi \right\}^{-1} z^{-\alpha_r}$$

$$\cdot\ E\left\{ \begin{matrix} \alpha_r, \alpha_r+\beta, -\alpha_{p+1}, \ldots, \alpha_r+\beta_\ell-\alpha_{p+1}, \alpha_r-\rho_1+1, \ldots, \alpha_r-\rho_q+1 : e^{\pm i\pi(p-q)} z \\ \alpha_r-\alpha_1+1, \ldots * \ldots, \alpha_r-\alpha_p+1, \alpha_r+\sigma_1-\alpha_{p+1}, \ldots, \alpha_r+\sigma_m-\alpha_{p+1} \end{matrix} \right\},$$

where $p \geq q+1$, $\ell \geq m+1$, $R(\alpha_{p+1}) > 0$, $R(\alpha_r+\beta_t-\alpha_{p+1}) > 0$, $r = 1,2,3,\ldots,p$; $t = 1,2,3,\ldots,\ell$ and $|\arg z| < \pi$, and

$$(24) \qquad \int_0^{\infty} x^{-\lambda-1} E(p;\alpha_r:q;\rho_s:\mu x)\, dx = \frac{\Gamma(\lambda) \prod_{r=1}^{p} \Gamma(\alpha_r-\lambda)}{\prod_{t=1}^{q} \Gamma(\rho_s-\lambda)} \mu^{\lambda}$$

where $R(\lambda) > 0$, $R(\alpha_r-\lambda) > 0$. To obtain the required expansion, multiply both sides of (14) by $x^{-\lambda}$ and integrate with respect to x, applying (23) and (24), thus obtaining

$$(25)\qquad \sum_{r=0}^{\infty} \frac{\Gamma(a-\lambda-r)\Gamma(b-\lambda-r)\Gamma(a+r)\Gamma(b'+r)\Gamma(\lambda+2r)}{r!\ \Gamma(a+r)}$$

$$\cdot\ {}_3F_2(a+2r,a+r,b'+r;1+\lambda-a+r,1+\lambda-b+r;-1)$$

$$+\sum_{r=0}^{\infty} \frac{\Gamma(\lambda-a+r)\Gamma(b-a)\Gamma(2a-\lambda)\Gamma(a+b'-\lambda)\Gamma(a+r)}{r!\ \Gamma(a+r)}$$

$$\cdot\ {}_3F_2(a+r,2a-\lambda,a+b'-\lambda;1+a-\lambda-r,1+a-b;-1)$$

$$+\sum_{r=0}^{\infty} \frac{\Gamma(a-b)\Gamma(\lambda+r-b)\Gamma(a+b-\lambda)\Gamma(b+b'-\lambda)\Gamma(b+r)}{r!\ \Gamma(a+r)}$$

$$\cdot\ {}_3F_2(b+r,a+b-\lambda,b+b'-\lambda;1+b-a,1+b-\lambda-r;-1)$$

$$=\Gamma(b)\Gamma(b')\Gamma(\lambda)\Gamma(a-\lambda)\Gamma(b+b'-\lambda)\left\{\Gamma(b+b')\right\}^{-1},$$

where $R(\lambda) > 0$, $R(b+b'-\lambda) > 0$, $R(2a-\lambda) > 0$, $R(a+b-\lambda) > 0$. It may be noted that some of the last restrictions can be removed by the theory of analytical continuation.

4. Duplication and addition formulas for the E-functions

The four formulas to be proved in this section are:

$$(26)\qquad \sum_{r=0}^{\infty} \frac{(-\frac{1}{2};r)\quad x^{-2r}}{r!\,\Gamma(a+r)\Gamma(b+r)}\left\{E(a+r,b+r::x)\right\}^{2} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(2a)\Gamma(2b)}\,E(2a,2b::2x),$$

which may be considered as the duplication formula for the E-function;

$$(27)\qquad \sum_{r=0}^{\infty} \frac{(\frac{1}{2};r)\ (4x)^{-2r}}{r!\,\Gamma(a+\frac{1}{2}+r)\Gamma(b+\frac{1}{2}+r)}\,E\left[2a+2r,2b+2r:2x\right] = \frac{2^{2a+2b-2}}{\pi\Gamma(a)\Gamma(b)}\left\{E(a,b::x)\right\}^{2},$$

(28)
$$\sum_{r=0}^{\infty} \frac{(-\frac{1}{2};r)\ x^{-2r}}{r!\,\Gamma(a+r)\Gamma(b+r)}\ E(a+r,b+r::x)E(a+1+r,b+r::x)$$

$$= \frac{\Gamma(a+1)\Gamma(b)}{\Gamma(2a+1)\Gamma(2b)}\ E(2a+1,2b::2x),$$

(29)
$$\sum_{r=0}^{\infty} \frac{(\frac{1}{2};r)(4x)^{-2r}}{r!\,\Gamma(a+\frac{1}{2}+r)\Gamma(b+\frac{1}{2}+r)}\ E(2a+2r+1,2b+2r::2x)$$

$$= \frac{2^{2a+2b-1}}{\pi\ \Gamma(a)\Gamma(b)}\ E(a,b::x)E(a+1,b::x).$$

To prove (26), consider the series

$$\sum_{r=0}^{\infty} \frac{(-\frac{1}{2};\ r)\ x^{-2r}}{r!\,\Gamma(a+r)\Gamma(b+r)}\ E(a+r,b+r::x)E(a+r,b'+r::x).$$

Now substitute from (15) for the product of the E-functions and get

$$\sum_{r=0}^{\infty} \frac{(-\frac{1}{2};r)\ x^{-2r}}{r!\,\Gamma(a+r)\Gamma(b+r)}\ E(a+r,b+r::x)E(a+r,b'+r::x)$$

$$= \sum_{r=0}^{\infty} \frac{(-\frac{1}{2};r)\ x^{-2r}}{r!\,\Gamma(b+r)} \sum_{s=0}^{\infty} \frac{(-1)^s\,\Gamma(b+r+s)\Gamma(b'+r+s)}{\Gamma(b+b'+2r+2s)}\ E(a+r+s,b+b'+2r+2s::x)$$

$$= \sum_{p=0}^{\infty} \frac{\Gamma(b+p)\Gamma(b'+p)}{p!\,\Gamma(b+b'+p)}\ x^{-2p}(-1)^p\ E(a+p,b+b'+2p::x)\ \frac{1}{\Gamma(b)}\ {}_2F_1\left(\begin{matrix}-1/2,-p\\ b\end{matrix}\right)$$

$$= \sum_{p=0}^{\infty} \frac{(-1)^p\,\Gamma(b'+p)x^{-2p}}{p!\,\Gamma(b+b'+2p)}\ \frac{\Gamma(b+\frac{1}{2}+p)}{\Gamma(b+\frac{1}{2})}\ E(a+p,b+b'+2p::x)$$

by means of summing by Gauss' theorem. Thus

$$\text{(A)} \qquad \sum_{r=0}^{\infty} \frac{(-\frac{1}{2}\,;r)\ \ x^{-2r}}{r!\,\Gamma(a+r)\Gamma(b+r)}\, E(a+r,b+r::x)E(a+r,b'+r::x)$$

$$= \sum_{p=0}^{\infty} \frac{(-1)^p\Gamma(b'+p)x^{-2p}}{p!\,\Gamma(b+b'+2p)}\ \frac{\Gamma(b+\frac{1}{2}+p)}{\Gamma(b+\frac{1}{2})}\, E(a+p,b'+b+2p::x).$$

In (A), take b' = b. The result is

$$\sum_{r=0}^{\infty} \frac{(-\frac{1}{2}\,;r)\ \ x^{-2r}}{r!\,\Gamma(a+r)\Gamma(b+r)} \left\{ E(a+r,b+r::x) \right\}^2 =$$

$$\frac{\sqrt{\pi}}{2^{2b-1}\Gamma(b+\frac{1}{2})} \sum_{p=0}^{\infty} \frac{(-1)^p(2x)^{-2p}}{p!}\, E(a+p,2b+2p::x)$$

$$= \frac{\Gamma(a)\Gamma(b)}{\Gamma(2a)\Gamma(2b)}\, E(2a,2b::2x), \quad \text{by (21).}$$

Thus (26) is proved. In (A) interchange a and b, then write (a+1) for b' and so obtain (28). To prove (27), substitute for the E-function on the left-hand side of (24) from (26) deriving

$$\text{l.h.s. of (27)} \qquad = \sum_{r=0}^{\infty} \frac{(\frac{1}{2}\,;r)\ (4x)^{-2r}}{r!\,\Gamma(a+\frac{1}{2}+r)\Gamma(b+\frac{1}{2}+r)}\ \frac{\Gamma(2a+2r)\Gamma(2b+2r)}{\Gamma(a+r)\ \Gamma(b+r)}$$

$$\cdot \sum_{s=0}^{\infty} \frac{(-\frac{1}{2}\,;s)x^{-2s}}{\Gamma(a+r+s)\Gamma(b+r+s)} \left\{ E(a+r+s,b+r+s::x) \right\}^2$$

$$= \sum_{p=0}^{\infty} \frac{2^{2a+2b-2}(-\frac{1}{2}\,;p)}{\pi\Gamma(a+p)\Gamma(b+p)}\, x^{-2p} \left\{ E(a+p,b+p::x) \right\}^2 \ {}_2F_1\begin{pmatrix} 1/2,-p;1 \\ 3/2-p \end{pmatrix} .$$

By Gauss' theorem, this ${}_2F_1$ can be summed as a product of gamma functions which includes $\Gamma(1-p)$ in the denominator. The ${}_2F_1$ thus vanishes unless $p = 0$. The outer sum therefore reduces to the first term

$$2^{2a+2b-2}\left\{\pi\Gamma(a)\Gamma(b)\right\}^{-1}\left\{E(a,b::x)\right\}^{2},$$

and the proof of (27) is complete. If we substitute from (28) in the left-hand side of (29), and proceed as before, formula (29) is obtained.

By substituting from (13) in (26), (28), (27), (29) respectively, formulas (5), (6), (7) and (8) are obtained.

5. A certain kind of transformation between generalized hypergeometric functions of arguments $+\frac{1}{2}$ and $+1$

Multiply both sides of equations (14), (26), (27), (28) and (29) by $e^{-x}x^{\gamma-1}$ and integrate with respect to x, applying (11) and (12), with the result that

$$(30)\qquad \sum_{r=0}^{\infty}\frac{(a;b)(b;r)(b';r)}{r!(a+b+b'+\gamma;r)\Gamma(2a+b'+\gamma+r)}$$

$$\cdot\ {}_3F_2\left[\begin{matrix}a+r,\ b'+r,\ a+b'+\gamma;\ 1\\ a+b+b'+\gamma+r,\ 2a+b'+\gamma+r\end{matrix}\right]=\frac{\Gamma(a+\gamma)}{\Gamma(2a+\gamma)\Gamma(a+b'+\gamma)},$$

$$(31)\qquad \sum_{r=0}^{\infty}\frac{(-\frac{1}{2};r)(a;r)(b;r)}{r!\Gamma(a+2b+\gamma+r)\Gamma(b+2a+\gamma+r)}\ {}_3F_2\left[\begin{matrix}a+r,b+r,a+b+\gamma;\ 1\\ a+2b+\gamma+r,b+2a+\gamma+r\end{matrix}\right]$$

$$=\left[\Gamma(a+b+\gamma)\Gamma(2a+2b+\gamma)\right]^{-1}\ {}_2F_1\left[\begin{matrix}2a,2b;\ \frac{1}{2}\\ 2a+2b+\gamma\end{matrix}\right],$$

$$(32)\qquad \sum_{r=0}^{\infty}\frac{(\frac{1}{2};r)(a;r)(b;r)}{r!\Gamma(2a+2b+\gamma+2r)}\ {}_2F_1\left[\begin{matrix}2a+2r,\ 2b+2r;\ 1/2\\ 2a+2b+\gamma+2r\end{matrix}\right]$$

$$=\frac{\Gamma(a+b+\gamma)}{\Gamma(a+2b+\gamma)\Gamma(b+2a+\gamma)}\ {}_3F_2\left[\begin{matrix}a,b,a+b+\gamma;1\\ a+2b+\gamma,\ b+2a+\gamma\end{matrix}\right],$$

(33) $$\sum_{r=0}^{\infty} \frac{(-\frac{1}{2};r)(a+1;r)(b;r)}{r!\,\Gamma(2a+b+\gamma+1+r)\,\Gamma(2b+a+\gamma+1+r)}$$

$$\cdot\ {}_3F_2\left[\begin{matrix}a+1+r, b+r, a+b+\gamma+1+r;1\\ 2a+b+\gamma+1+r, 2b+a+\gamma+1+r\end{matrix}\right]$$

$$= \frac{1}{\Gamma(a+b+\gamma+1)\,\Gamma(2a+2b+\gamma+1)}\ {}_2F_1\left[\begin{matrix}2a+1;2b;1/2\\ 2a+2b+\gamma+1\end{matrix}\right],$$

and

(34) $$\sum_{r=0}^{\infty} \frac{(\frac{1}{2};r)(a+1;r)(b;r)}{r!\,\Gamma(2a+2b+\gamma+1+2r)}\ {}_2F_1\left[\begin{matrix}2a+2r+1, 2b+2r;1/2\\ 2a+2b+\gamma+1+2r\end{matrix}\right]$$

$$= \frac{\Gamma(a+b+\gamma+1)}{\Gamma(2a+b+\gamma+1)\,\Gamma(2b+a+\gamma+1)}\ {}_3F_2\left[\begin{matrix}a+1, b, a+b+\gamma+1;\ 1\\ 2a+b+\gamma+1, 2b+a+\gamma+1\end{matrix}\right].$$

[1] MacRobert, T.M. - Functions of a Complex Variable; New York, MacMillan, 1954, 4th edition.

[2] Ragab, F.M. - Proceedings Glasgow Mathematical Association; 3, 1957, (in press).

[3] Titchmarsh, E.C. - Theory of Fourier Integrals; London, Oxford Univ. Press, 1937.

[4] Ragab, F.M. - Proceedings Glasgow Mathematical Association; 1, 1953.

DISTRIBUTION LIST FOR RESEARCH REPORTS

Contract No. AF 18(600)-367

(ONE copy unless otherwise noted)

Commandant
Headquarters Air University
Maxwell Air Force Base, Alabama

(3)Commander
European Office, ARDC
60 Rue Ravenstein
Brussels, Belgium

Applied Mathematics & Statistics Lab.
Stanford University
Stanford, California

Department of Mathematics
University of California
Berkeley, California

Commander
Air Force Flight Test Center
ATTN: Technical Library
Edwards Air Force Base, California

The Rand Corporation
Technical Library
1700 Main Street
Santa Monica, California

Director of Advanced Studies
Air Force Office of Scientific Research
Post Office Box 2035
Pasadena 2, California

Commander
Western Development Division
ATTN: WDSIT
P.O. Box 262
Inglewood, California

Department of Mathematics
Yale University
New Haven, Connecticut

Chief of Naval Operations
Department of the Navy
ATTN: OP03EG; Mr. Arthur Kaufman
Washington 25, D.C.

Human Factors Operations Research Labs.
Air Research and Development Command
Bolling Air Force Base
Washington 25, D.C.

(2)Office of Naval Research
Department of the Navy
ATTN: Code 432
Washington 25, D.C.

Department of Commerce
Office of Technical Services
Washington 25, D.C.

Director National Security Agency
ATTN: Dr. H.H. Campaigne
Washington 25, D.C.

Library
National Bureau of Standards
Washington 25, D.C.

National Applied Mathematics Labs.
National Bureau of Standards
Washington 25, D.C.

Headquarters, USAF
Director of Operations
ATTN: Operations Analysis Division, AFOOP
Washington 25, D.C.

(2)Commander
Air Force Office of Scientific Research
ATTN: SRDB
Washington 25, D.C.

(2)Commander
Air Force Office of Scientific Research
ATTN: SRE
Washington 25, D.C.

National Science Foundation
1520 "H" St., N.W.
Washington 25, D.C.

Commander
Air Force Armament Center
ATTN: Technical Library
Eglin Air Force Base, Florida

Commander
Air Force Missile Test Center
ATTN: Technical Library
Patrick Air Force Base, Florida

Department of Mathematics
Northwestern University
Evanston, Illinois

Institute for Air Weapons Research
Museum of Science and Industry
University of Chicago
Chicago 37, Illinois

Department of Mathematics
University of Chicago
Chicago 37, Illinois

Department of Mathematics
University of Illinois
Urbana, Illinois

Department of Mathematics
Purdue University
Lafayette, Indiana

Institute for Fluid Dynamics
and Applied Mathematics
University of Maryland
College Park, Maryland

Mathematics and Physics Library
The Johns Hopkins University
Baltimore, Maryland

Commander
HQ Air Research and Development Command
ATTN: Technical Library
P.O. Box 1395
Baltimore 3, Maryland

Department of Mathematics
Harvard University
Cambridge 38, Massachusetts

Department of Mathematics
Massachusetts Institute of Technology
Cambridge, Massachusetts

Commander
Air Force Cambridge Research Center
ATTN: Geophysics Research Library
L.G. Hanscom Field
Bedford, Massachusetts

Commander
Air Force Cambridge Research Center
ATTN: Electronics Research Library
L.G. Hanscom Field
Bedford, Massachusetts

Dr. Y.W. Chen
2645 Stuart Street
Berkeley, California

Department of Mathematics
Wayne University
Detroit 1, Michigan

Willow Run Research Center
University of Michigan
Ypsilanti, Michigan

Department of Mathematics
Folwell Hall
University of Minnesota
Minneapolis, Minnesota

Department of Mathematics
Institute of Technology
Engineering Building
University of Minnesota
Minneapolis, Minnesota

Department of Mathematics
Washington University
St. Louis 5, Missouri

Department of Mathematics
University of Missouri
Columbia, Missouri

Linda Hall Library
ATTN: Mr. Thomas Gillis
Document Division
5109 Cherry Street
Kansas City 10, Missouri

Commander
Strategic Air Command
ATTN: Operations Analysis
Offutt Air Force Base
Omaha, Nebraska

The James Forrestal Res. Center Lib.
Princeton University
Princeton, New Jersey

Library
Institute for Advanced Study
Princeton, New Jersey

Department of Mathematics
Fine Hall
Princeton University
Princeton, New Jersey

Commander
Holloman Air Development Center
ATTN: Technical Library
Holloman Air Force Base, New Mexico

Commander
Air Force Special Weapons Center
ATTN: Technical Library
Kirtland Air Force Base
Albuquerque, New Mexico

Prof. J. Wolfowitz
Mathematics Department
White Hall
Cornell University
Ithaca, New York

Department of Mathematics
Syracuse University
Syracuse, New York

New York University
Institute of Mathematical Sciences
Division of Electromagnetic Research
ATTN: Professor M. Kline
25 Waverly Place
New York 3, N.Y.

Department of Mathematics
Columbia University
ATTN: Professor B.O. Koopman
New York 27, N.Y.

Department of Mathematical Statistics
Fayerweather Hall
ATTN: Dr. Herbert Robbins
Columbia University
New York 27, N.Y.

Commander
Rome Air Development Center
ATTN: Technical Library
Griffiss Air Force Base
Rome, New York

Institute for Aeronautical Sciences
ATTN: Librarian
2 East 64th Street
New York 21, N.Y.

Institute of Statistics
North Carolina State College of A& E
Raleigh, North Carolina

Department of Mathematics
University of North Carolina
Chapel Hill, North Carolina

(2)Office of Ordnance Research
Box CM
Duke Station
Durham, North Carolina

Department of Mathematics
Duke University
Duke Station
Durham, North Carolina

Commander
Air Technical Intelligence Center
ATTN: ATIAE-4
Wright-Patterson Air Force Base, Ohio

Commander
Wright Air Development Center
ATTN: Technical Library
Wright-Patterson Air Force Base, Ohio

(2)Commander
Wright Air Development Center
ATTN: ARL Technical Library, WCRR
Wright-Patterson Air Force Base, Ohio

(2)Commandant
USAF Institute of Technology
ATTN: Technical Library, MCLI
Wright-Patterson Air Force Base, Ohio

(10)Chief, Document Service Center
Armed Services Technical Information Agency
Knott Building
Dayton 2, Ohio

Department of Mathematics
Carnegie Institute of Technology
Pittsburgh, Pennsylvania

Department of Mathematics
University of Pennsylvania
Philadelphia, Pennsylvania

Commander
Arnold Engineering Development Center
ATTN: Technical Library
Tullahoma, Tennessee

Defense Research Laboratory
University of Texas
Austin, Texas

Department of Mathematics
Rice Institute
Houston, Texas

Commander
AF Personnel & Training Research Center
ATTN: Technical Library
Lackland Air Force Base
San Antonio, Texas

Department of Mathematics
University of Wisconsin
Madison, Wisconsin

Mathematics Research Center
ATTN: R.E. Langer
University of Wisconsin
Madison, Wisconsin

Naval Electronics Laboratory
San Diego 52, California

Research Laboratory of Electronics
Massachusetts Institute of Technology
Rm. 20B-221
Cambridge 39, Massachusetts

Signal Corps Engineering Labs.
Technical Documents Center
Evans Signal Laboratory
Belmar, New Jersey

Commanding General
Signal Corps Engineering Labs.
ATTN: Technical Reports Library
Fort Monmouth, New Jersey

Chief of Staff
HQ United States Air Force
Pentagon, Washington 25, D.C.
ATTN: AFMRD - 5

Office of Chief Signal Officer
Signal Plans and Operations Division
Com. Liaison Br., Radio Prop. Section
The Pentagon, Washington 25, D.C.
ATTN: SIGOL - 2, Rm. 20

Library
Boulder Laboratory
National Bureau of Standards
Boulder, Colorado

Naval Research Laboratory
Washington 25, D.C.
ATTN: Technical Data Section

Air Force Development Field Rep.
Naval Research Laboratory
Code 1110
Washington 25, D.C.

Ballistics Research Laboratory
Aberdeen Proving Ground
Aberdeen, Maryland
ATTN: Dr. Keats Pullen

Massachusetts Institute of Technology
Lincoln Laboratory
P.O. Box 390
Cambridge 39, Massachusetts
ATTN: Dr. T.J. Carroll

Mr. Keeve Siegel
Willow Run Research Center
Willow Run Airport
Ypsilanti, Michigan

Willow Run Research Center
University of Michigan
Willow Run Airport
Ypsilanti, Michigan
ATTN: Dr. C.L. Dolph

Director
Naval Research Laboratory
Code 3480
Washington 25, D.C.
ATTN: Dr. L.C. van Atta

Director
Naval Research Laboratory
Washington 25, D.C.
ATTN: Mr. Robert E. Roberson

National Bureau of Standards
Computation Laboratories
Washington 25, D.C.
ATTN: Dr. John Todd

Dr. A.G. McNish
National Bureau of Standards
Washington 25, D.C.

Dr. Derrick H. Lehmer
Department of Mathematics
University of California
Berkeley, California

Division of Electrical Engineering
Electronics Research Laboratory
University of California
Berkeley 4, California
ATTN: Dr. Samuel Silver

Dr. Joseph Kaplan
Department of Physics
University of California
Los Angeles, California

Dr. A. Erdelyi
California Institute of Technology
1201 E. California Street
Pasadena, California

Dr. Robert Kalaba
Electronics Division
Rand Corporation
Santa Monica, California

Stanford Research Institute
Stanford, California
ATTN: Dr. J.V.N. Granger
Head, Radio Systems Laboratory

Dr. Vic Twersky
Electronics Defense Laboratory
Box 205
Mountain View, California

Hughes Aircraft Company
Research and Development Library
Culver City, California
ATTN: M. Bodner

Georgia Institute of Technology
Engineering Experimental Station
Atlanta, Georgia
ATTN: Dr. James E. Boyd

Applied Physics Laboratory
The Johns Hopkins University
8621 Georgia Avenue
Silver Spring, Maryland
ATTN: Mr. F.T. McClure

Dr. Donald E. Kerr
Department of Physics
The Johns Hopkins University
Baltimore 18, Maryland

Dr. A. Weinstein
Institute of Fluid Dynamics and
Applied Mathematics
University of Maryland
College Park, Maryland

Professor J.A. Pierce
Harvard University
Cambridge 38, Massachusetts

Dr. F.M. Wiener
Bolt Beranek and Newman Inc.
16 Eliot Street
Cambridge 38, Massachusetts

Mrs. Marjorie L. Cox, Librarian
Technical Reports Collection
Room 303A, Pierce Hall
Harvard University
Cambridge 38, Massachusetts

Dr. Arthur A. Oliner
Microwave Research Institute
Polytechnic Institute of Brooklyn
55 Johnson Street
Brooklyn, N.Y.

Professor J.H. Mulligan
School of Engineering
New York University
New York, N.Y.

Dr. H.G. Booker
School of Electrical Engineering
Cornell University
Ithaca, N.Y.

Professor Bernard Epstein
Department of Mathematics
University of Pennsylvania
Philadelphia 4, Pennsylvania

Professor Albert Heins
Carnegie Institute of Technology
Pittsburgh, Pennsylvania

Professor Fred A. Ficken
University of Tennessee
Knoxville, Tennessee

Miss Barbara C. Grimes, Librarian
Federal Communications Commission
Washington 25, D.C.

Carnegie Institute of Washington
Department of Terrestrial Magnetism
5241 Broad Branch Road, N.W.
Washington 15, D.C.
ATTN: Library

Mathematical Reviews
80 Waterman Street
Brown University
Providence, Rhode Island

Mr. Martin Katzin
154 Fleetwood Terrace
Silver Spring, Maryland

Prof. B.H. Bissinger
Lebanon Valley College
Annville, Pennsylvania

Applied Physics Laboratory
The Johns Hopkins University
8621 Georgia Avenue
Silver Spring, Maryland
ATTN: Dr. B.S. Gourary

Dr. Charles H. Papas
Dept. of Electrical Engineering
California Institute of Technology
Pasadena, California

Dr. Rodman Doll
209A Emmet Street
Ypsilanti, Michigan

California Institute of Technology
Pasadena 4, California
ATTN: Mr. Calvin Wilcox

Technical Director
Combat Development Dept.
Army Electronic Proving Ground
Fort Huachuca, Arizona

National Bureau of Standards
Boulder, Colorado
ATTN: Dr. W.R. Callet

Mrs. A.M. Gray
Engineering Library, Plant 5
Grumman Aircraft Corp.
Bethpage, L.I., N.Y.

Professor Bernard Friedman
55 Hilltop Avenue
New Rochelle, N.Y.

Dr. Jane Scanlon
224-03 67th Avenue
Bayside 64, N.Y.

Dr. Solomon L. Schwebel
3689 Louis Road
Palo Alto, California

Dr. Bernard Lippmann
Eastern Technical Associates
38 Main Street
Hastings-On-Hudson, N.Y.

Engineering Library
University of California
405 Hilgard Avenue
Los Angeles 24, California

University of Minnesota
The University Library
Minneapolis 14, Minnesota
ATTN: Exchange Division

Lincoln Laboratory
Massachusette Institute of Technology
P.O. Box 73
Lexington 73, Massachusetts
ATTN: Dr. Shou Chin Wang, Rm. C-351

Technical Research Group
17 Union Square West
New York 3, N.Y.
ATTN: Dr. L. Goldmuntz

Hoffman Laboratories Inc.
Advanced Development Section
3761 South Hill Street
Los Angeles 7, California
ATTN: Dr. Richard B. Barrar

Institute of Fluid Dynamics and
Applied Mathematics
University of Maryland
College Park, Maryland
ATTN: Dr. Elliott Montroll

Brandeis University
Waltham, Massachusetts
ATTN: Library

General Electric Company
Microwave Laboratory
Electronics Division
Stanford Industrial Park
Palo Alto, California
ATTN: Library

Dr. John B. Smyth
Smyth Research Associates
3930 4th Avenue
San Diego 3, California

Dr. Georges G. Weill
Electrical Engineering
California Institute of Technology
Pasadena, California

Naval Research Laboratory
Washington 25, D.C.
ATTN: Dr. Henry J. Passerini
Code 5278 A

Dr. George Kear
10585 N. Stelling Road
Cupertino, California

California Institute of Technology
Electrical Engineering
Pasadena, California
ATTN: Dr. Zohrab A. Kaprielian

Prof. Nathan Marcuvitz
Microwave Research Institute
Brooklyn Polytechnic
55 Johnson Street
Brooklyn, N.Y.

Dr. Lester Kraus
4935 WhitehavenWay
San Diego 10, California

Dr. Jerry Shmoys
Dept. of Electrical Engineering
Brooklyn Polytechnic
85 Livingston Street
Brooklyn, N.Y.

Dr. I. Kolodner
Dept. of Mathematics
University of New Mexico
Albuquerque, New Mexico

Dr. Harry Hochstadt
W.L. Maxon
460 W. 34th Street
New York, N.Y.

Hycon Eastern Inc.
75 Cambridge Parkway
Cambridge, Massachusetts
ATTN: Seymour Stein

Mr. K.S. Kelleher, Section Head
Melpar, Inc.
3000 Arlington Boulevard
Falls Church, Virginia

Department of Engineering
Brown University
Providence, R.I.
ATTN: Dr. V.M. Papadopoulos

Professor C.A. Woonton, Director
Eaton Electronics Research Laboratory
McGill University
Montreal, Canada

Dr. D.S. Jones
University of North Staffordshire
Keele, Staffordshire
England

Science Abstracts
Institute of Electrical Engineers
Savoy Place
London W.C. 2, England
ATTN: R.M. Crowther

Dr. Jean-Claude Simon
Centre de Recherches techniques
Compagnie generale de T.S.F.
Paris 19, France

Professor A. van Wijngaarden
Mathematisch Centrum
2e Boerhaavestraat 49
Amsterdam-Zuid, Holland

Dr. C.J. Bouwkamp
Philips Research Laboratories
Eindhoven, Netherlands

Dr. E.T. Copson
Department of Mathematics
United College
University of St. Andrews
St. Andrews, Scotland

Universite de Paris
Cabinet du Department
des Sciences Mathematiques
Institut Henri Poincare
11 Rue Pierre Curie
Paris 5e, France

Dr. W. Elwyn Williams
24 Manton Drive
Luton
Bedfordshire, England

University of Cambridge
Cavendish Laboratory
Cambridge, England
ATTN: Prof. Philip Clemmow

Technische Hogeschool
Instituut voor Toegepaste Wiskunde
Jaffalaan 162
Delft, Holland
ATTN: Prof. R. Timman

Instituto de Fisica Teorica
Rue Pamplona, 145
S. Paulo, Brasil

Dr. H. Bremmer
Philips Research Laboratories
Eindhoven, Netherlands

Electrotechnical Laboratory
Nagata-cho
Chiyoda-ku
Tokyo, Japan
ATTN: Library

Date Due

NOV 24 '59

PRINTED IN U. S. A.

NYU
BR-
23 Ragab.
c. 1 Expansions for products
of two Whittaker functions.

NYU
BR-
23 Ragab.
AUTHOR
c. 1 Expansions for products
TITLE
of two Whittaker functions

DATE DUE	BORROWER'S NAME	ROOM NUMBER
~~NOV 24 [illegible]~~		

N. Y. U. Institute of
Mathematical Sciences
25 Waverly Place
New York 3, N. Y.